Lazy categories

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Introduction

- Motivated by certain problems in the theory of double categories
- (Strict) double categories are categories in $\mathbf{Cat}$
- Many properties of $\mathbf{Doub}$ follow from the general theory of $\mathbf{Cat}(\mathbf{C})$
- **Example:** $\mathbf{Cat}(\mathbf{C})$ has a 2-category structure
- **Example:** $\mathbf{C}$ cartesian closed with finite limits, then so is $\mathbf{Cat}(\mathbf{C})$
- When proving such results, we secretly pretend that $\mathbf{C}$ is $\mathbf{Set}$

Cat is not Set

Cat has many nice set-like properties

- Complete and cocomplete
- Locally finitely presentable
- Cartesian closed
- Sums are disjoint and universal

But

- Not a regular category

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & \mathbf{2} \arrow[r, two heads] & \mathbb{N} \\
& \mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1} \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxed{\text{Pb}}"] & \mathbf{2} \arrow[u, "2"']
\end{tikzcd}
$$

- Not locally cartesian closed

Culprit: Quotients
Plan: Fix quotients and not break anything else

Fixing quotients

- Embed $\mathbf{Cat}$ in a suitable Grothendieck topos
- Introduced by Verity in his thesis to study double limits
- $\mathbf{Cat} \hookrightarrow \mathbf{sSet}$
- Other possibilities, e.g. $\mathbf{Set}^{\mathbf{fpCat}^{op}}$
- Smallest one is sheaves on the canonical topology on $\mathbf{Cat}$

The topos hull of $\mathbf{Cat}$

Theorem

(1) *$\langle F_i : \mathbf{C}_i \longrightarrow \mathbf{C} \rangle_{i \in I}$ is a covering family for the canonical topology on $\mathbf{Cat}$ if and only if it is onto on composable pairs.*

(2) *The category of sheaves on $\mathbf{Cat}$ for the canonical topology is equivalent to*

$$\mathbf{Set}^{\Delta_2^{op}},$$

2-truncated simplicial sets.

(3) *It is also equivalent to M-$\mathbf{Set}$ (right M-sets) where M is the monoid of endomorphisms of $\mathbf{3}$*

$\mathbf{Set}^{\Delta_2^{op}}$ is like $\mathbf{Set}$ but it's also like $\mathbf{Cat}$!

Lazy categories

Definition

A *lazy category* is an object of $\mathbf{Set}^{\Delta_2^{op}}$

$$
\mathbf{C} \quad = \quad \begin{tikzcd} C_2 \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & C_1 \arrow[l, shift right=2, "\lambda"'] \arrow[l, shift left=2, "\rho"] \arrow[r, shift left=2] \arrow[r, shift right=2] & C_0 \arrow[l, "\mathrm{id}"'] \end{tikzcd}
$$

- The elements of C_0 are called *objects*, denoted $A, A', B, \ldots$
- The elements of C_1 are called *morphisms* (sometimes *idle or lazy morphisms*) and denoted $f\colon A \longrightarrow B, \ldots$
- The elements of C_2 are called *composers* and denoted

$$
\begin{tikzcd} & A_1 \arrow[dr, "a_1"] & \\ A_0 \arrow[ur, "a_0"] \arrow[rr, "a_2"'] \arrow[rr, phantom, shift left=5, "\alpha"] & & A_0 \end{tikzcd}
$$

Identities

- $\mathrm{id}(A)$ is denoted $1_A \colon A \longrightarrow A$ and called the *identity morphism* on A
- $\lambda(f)$ is denoted by

$$\begin{tikzcd} & B \arrow[dr, "1_B"] \arrow[d, phantom, "\lambda_f"] & \\ A \arrow[ur, "f"] \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

  and called the *left identity composer*
- $\rho(f)$ is denoted by

$$\begin{tikzcd} & A \arrow[dr, "f"] \arrow[d, phantom, "\rho_f"] & \\ A \arrow[ur, "1_A"] \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

  and called the *right identity composer*

All of the equations that these satisfy are subsumed by the notation except for $\lambda_{1_A} = \rho_{1_A}$ which we write as ι_A.

- A *lazy functor* $F \colon \mathbf{A} \longrightarrow \mathbf{X}$ is required to preserve all of the above. We write the category of lazy categories and lazy functors as $\mathbf{LCat}$.

Properties of $\mathbf{LCat}$

Theorem

(1) $\mathbf{Cat}$ *is a reflective subcategory of* $\mathbf{LCat}$*.*

(2) $\mathbf{Cat} \rightarrowtail \mathbf{LCat}$ *creates all limits.*

(3) $\mathbf{Cat} \rightarrowtail \mathbf{LCat}$ *creates coproducts and filtered colimits.*

(4) $\mathbf{Cat} \rightarrowtail \mathbf{LCat}$ *preserves exponentiation, in fact* $\mathbf{Cat}$ *is an exponential ideal in* $\mathbf{LCat}$*.*

(5) *The inclusion* $\mathbf{Cat} \rightarrowtail \mathbf{LCat}$ *preserves quotients by congruences.*

Theorem

(1) $\mathbf{LCat}$ *is monadic over* $\mathbf{Set}$*. It is algebras for the monad induced by the composable pairs functor* $P\colon \mathbf{Cat} \longrightarrow \mathbf{Set}$ *and its left adjoint*

$$\begin{tikzcd} \mathbf{Cat} \arrow[rr, tail] \arrow[dr, "P"'] & & \mathbf{LCat} \arrow[dl, "P"] \\ & \mathbf{Set} & \end{tikzcd}$$

(2) *Every lazy category* $\mathbf{A}$ *is a quotient of categories* $\mathbf{X}, \mathbf{Y}$

$$\mathbf{Y} \rightrightarrows \mathbf{X} \twoheadrightarrow \mathbf{A}.$$

Some lazy category theory

The internal hom

- For $\mathbf{A}$ and $\mathbf{X}$ lazy categories, we have $\mathbf{X}^{\mathbf{A}}$
- Objects of $\mathbf{X}^{\mathbf{A}}$ are (lazy) functors $\mathbf{A} \longrightarrow \mathbf{X}$
- The morphisms, which we call *(lazy) natural transformations*, can be determined by analyzing
$$\frac{\mathbf{2} \longrightarrow \mathbf{X}^{\mathbf{A}}}{\mathbf{A} \times \mathbf{2} \longrightarrow \mathbf{X}}$$
- The composers are determined by
$$\frac{\mathbf{3} \longrightarrow \mathbf{X}^{\mathbf{A}}}{\mathbf{A} \times \mathbf{3} \longrightarrow \mathbf{X}}$$

Lazy natural transformations

For $F, G\colon \mathbf{A} \longrightarrow \mathbf{X}$ lazy functors, a lazy natural transformation $t\colon F \longrightarrow G$ consists of:

- For every morphism $f\colon A \longrightarrow B$ in $\mathbf{A}$ a morphism

$$tf\colon FA \longrightarrow GB$$

  in $\mathbf{X}$
- For every composer in $\mathbf{A}$

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=4ex, "\alpha"] & & C \end{tikzcd}$$

  two composers in $\mathbf{X}$ (*left* and *right naturalizers*) $t_\lambda\alpha$ and $t_\rho\alpha$

$$\begin{tikzcd} FA \arrow[r, "tf"] \arrow[d, "Ff"'] \arrow[dr, "th" description] \arrow[dr, phantom, shift left=3ex, "t_\lambda\alpha"] \arrow[dr, phantom, shift right=3ex, "t_\rho\alpha"'] & GB \arrow[d, "Gg"] \\ FB \arrow[r, "tg"'] & GC \end{tikzcd}$$

subject to the conditions

$$t_\lambda(\lambda_f) = \lambda_{tf} \text{ and } t_\rho(\rho_f) = \rho_{tf}.$$

Natural composers

For lazy functors F, G, H and lazy natural transformations t, u, v as below, a *natural composer* ϕ

$$\begin{tikzcd} & G \arrow[dr, "u"] \arrow[d, phantom, "\phi"] & \\ F \arrow[ur, "t"] \arrow[rr, "v"'] & {} & H \end{tikzcd}$$

is given by:
For every composer α in $\mathbf{A}$ a composer $\phi(\alpha)$ in $\mathbf{X}$

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd} \quad \overset{\phi}{\longmapsto} \quad \begin{tikzcd} & GB \arrow[dr, "ug"] \arrow[d, phantom, "\phi\alpha"] & \\ FA \arrow[ur, "tf"] \arrow[rr, "vh"'] & {} & HC \end{tikzcd}$$

subject to no extra conditions.

- When $\mathbf{A}$ and $\mathbf{X}$ are (actual) categories, lazy functors and natural transformations reduce to the usual thing, and composers are uniquely determined by vertical composition of natural transformations.

Horizontal composition

As $\mathbf{LCat}$ is cartesian closed it's enriched over itself giving composition morphism

$$\mathbf{LCat}(\mathbf{B},\mathbf{C}) \times \mathbf{LCat}(\mathbf{A},\mathbf{B}) \longrightarrow \mathbf{LCat}(\mathbf{A},\mathbf{C})$$

- Associative and unitary.
- Composition of (lazy) functors is just function composition.
- Composition of (lazy) natural transformations is horizontal composition

$$\begin{tikzcd} \mathbf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow t"] & \mathbf{X} \arrow[r, bend left, "Q"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\Downarrow q"] & \mathbf{W} \end{tikzcd}$$

$$(qt)(f) = q(tf)$$

$$(A \xrightarrow{f} B) \quad \overset{t}{\longmapsto} \quad (FA \xrightarrow{tf} GB) \quad \overset{q}{\longmapsto} \quad (QFA \xrightarrow{qtf} RGB).$$

- Associative and unitary.
- Gives whiskering Qt and qF.

Composers compose

Composers also compose horizontally

$$\begin{tikzcd}[column sep=huge]
\mathbf{A} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=50, "H"'{name=H}] & \mathbf{X} \arrow[r, bend left=50, "Q"{name=Q}] \arrow[r, "R"{name=R, description}] \arrow[r, bend right=50, "S"'{name=S}] & \mathbf{W}
\arrow[Rightarrow, from=F, to=G, "t"]
\arrow[Rightarrow, from=G, to=H, "u"]
\arrow[Rightarrow, from=Q, to=R, "q"]
\arrow[Rightarrow, from=R, to=S, "r"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & G \arrow[dr, "u"] \arrow[d, phantom, "\phi"] & \\
F \arrow[ur, "t"] \arrow[rr, "v"'] & {} & H
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \arrow[dr, "r"] \arrow[d, phantom, "\theta"] & \\
Q \arrow[ur, "q"] \arrow[rr, "s"'] & {} & S
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\
A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C
\end{tikzcd}
\mapsto
\begin{tikzcd}
 & GB \arrow[dr, "ug"] \arrow[dl, "tf"'] \arrow[d, phantom, "\phi\alpha"] & \\
FA \arrow[rr, "vh"'] & {} & HC
\end{tikzcd}
\mapsto
\begin{tikzcd}
 & RGB \arrow[dr, "(bu)g"] \arrow[d, phantom, "\theta\phi\alpha"] & \\
QFA \arrow[ur, "(qt)f"] \arrow[rr, "(sv)h"'] & {} & SHC
\end{tikzcd}$$

- This is lazy interchange.

Comma categories

For F and G as below we have $(F \downarrow G)$

$$
\begin{tikzcd}
 & \mathbf{B} \arrow[dr, "F"] & \\
(F \downarrow G) \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow\kappa"] & & \mathbf{A} \\
 & \mathbf{C} \arrow[ur, "G"'] &
\end{tikzcd}
$$

- Objects $(B, a\colon FB \longrightarrow GC, C)$
- Morphisms

$$B \xrightarrow{b} B'$$

$$
\begin{tikzcd}
FB \arrow[r, "Fb"] \arrow[d, "a"'] \arrow[dr, "\overline{a}" description] & FB' \arrow[d, "a'"] \arrow[dl, phantom, "\alpha_\lambda" near start] \\
GC \arrow[r, "Gc"'] \arrow[ur, phantom, "\alpha_\rho" near start] & GC'
\end{tikzcd}
$$

$$C \xrightarrow[c]{} C'$$

– Composers

$$
\begin{tikzcd}
& (B',a',C') \arrow[dr, "{(b',\overline{a'},\alpha',c')}"] & \\
(B,a,C) \arrow[ur, "{(b,\overline{a},\alpha,c)}"] \arrow[rr, "{(b'',\overline{a''},\alpha'',c'')}"'] \arrow[rr, phantom, shift left=4ex, "{(\beta,\phi,\gamma)}"] & & (B'',a'',C'')
\end{tikzcd}
$$

$$
\begin{tikzcd}
& B' \arrow[dr, "b'"] & \\
B \arrow[ur, "b"] \arrow[rr, "b''"'] \arrow[rr, phantom, shift left=4ex, "\beta"] & & B''
\end{tikzcd}
\qquad
\begin{tikzcd}
FB \arrow[r, "Fb"] \arrow[d, "\overline{a}"'] \arrow[dr, "\overline{a'}" description] & FB' \arrow[d, "\overline{a'}"] \arrow[dl, phantom, "\phi_\lambda", shift left=3ex] \\
GC \arrow[r, "Gc"'] \arrow[ur, phantom, "\phi_\rho", shift right=3ex] & GC''
\end{tikzcd}
\qquad
\begin{tikzcd}
& C' \arrow[dr, "c'"] & \\
C \arrow[ur, "c"] \arrow[rr, "c''"'] \arrow[rr, phantom, shift left=4ex, "\gamma"] & & C''
\end{tikzcd}
$$

Lazy comma categories satisfy the usual universal property.

Lazy adjoints

- A lazy adjunction $F \dashv U \colon \mathbf{B} \longrightarrow \mathbf{A}$ is given by bijections $\phi, \phi_\lambda, \phi_\rho$
  -
$$(FA \xrightarrow{b} B) \quad \longleftrightarrow \quad (A \xrightarrow{\phi b} UB)$$
  -
$$\begin{tikzcd}
 & B \arrow[dd, "b'"] & & & UB \arrow[dd, "Ub'"] \\
FA \arrow[ur, "b"] \arrow[dr, "b''"'] \arrow[r, phantom, "\beta"] & & \longleftrightarrow & A \arrow[ur, "\phi b"] \arrow[dr, "\phi b''"'] \arrow[r, phantom, "\phi_\lambda \beta"] & \\
 & B' & & & UB'
\end{tikzcd}$$

$$\begin{tikzcd}
FA \arrow[dd, "Fa"'] \arrow[dr, "b"] & & & A \arrow[dd, "a"'] \arrow[dr, "\phi b"] & \\
\beta & B & \longleftrightarrow & \phi_\rho \beta & UB \\
FA' \arrow[ur, "b'"'] & & & A' \arrow[ur, "\phi b'"'] &
\end{tikzcd}$$

  satisfying
$$\phi_\lambda(\lambda_b) = \lambda_{\phi b} \quad \text{and} \quad \phi_\rho(\rho_b) = \rho_{\phi b}.$$

Example

$\Delta \colon \mathbf{A} \longrightarrow \mathbf{A^2}$ is a left adjoint to $\partial_0 \colon \mathbf{A^2} \longrightarrow \mathbf{A}$ if and only if $\mathbf{A}$ is a category.

Example

$\mathbf{A} \longrightarrow \mathbf{1}$ has a right adjoint if and only if there is an object $\top$ such that for every A there is a unique morphism

$$! \colon A \longrightarrow \top$$

and for every morphism $f \colon A \longrightarrow B$ there is a unique composer

$$\begin{tikzcd} A \arrow[dr, "!"] \arrow[dd, "f"'] & \\ & \top \\ B \arrow[ur, "!"'] & \end{tikzcd}$$

$$!$$

Theorem

(1) *Adjoints compose*

(2) *Adjoints are "lazily unique"*

(3) *Adjoints are Lawvere adjoints*

(4) *Adjoints are Street adjoints*

- F *is a* Lawvere left adjoint *to* U *if there is an isomorphism over* $\mathbf{A} \times \mathbf{B}$

$$\begin{tikzcd} (F \downarrow \mathbf{B}) \arrow[rr, "L", "\cong"'] \arrow[dr] & & (\mathbf{A} \downarrow U) \arrow[dl] \\ & \mathbf{A} \times \mathbf{B} & \end{tikzcd}$$

- F *is a* Street left adjoint *to* U *if there are natural transformations* $h\colon 1_{\mathbf{A}} \longrightarrow UF$ *and* $e\colon FU \longrightarrow 1_{\mathbf{B}}$ *and two composers*

$$\begin{tikzcd} & UFU \arrow[dr, "Ue"] \arrow[d, phantom, "\eta"] & \\ U \arrow[ur, "hU"] \arrow[rr, "1_U"'] & {} & U \end{tikzcd} \qquad \begin{tikzcd} & FUF \arrow[dr, "eF"] \arrow[d, phantom, "\epsilon"] & \\ F \arrow[ur, "Fh"] \arrow[rr, "1_F"'] & {} & F \end{tikzcd}$$

Lazy profunctors

Definition

A *lazy profunctor* $P\colon \mathbf{A} \relbar\!\!\bullet\!\!\rightarrow \mathbf{B}$ consists of

1. For every A in $\mathbf{A}$ and B in $\mathbf{B}$ a set $P(A,B)$ of *exomorphisms* $x\colon A \relbar\!\!\bullet\!\!\rightarrow B$
2. For every $a\colon A' \longrightarrow A$ in $\mathbf{A}$ and B in $\mathbf{B}$ a set $P(a,B)$ of *right actors*

$$\begin{tikzcd}
A \arrow[dr, "x", "\bullet" marking] & \\
\arrow[r, phantom, "\xi"] & B \\
A' \arrow[uu, "a"] \arrow[ur, "x'"', "\bullet" marking] &
\end{tikzcd}$$

3. For every A in $\mathbf{A}$ and $b\colon B \longrightarrow B$ in $\mathbf{B}$ a set $P(A,b)$ of *left actors*

$$\begin{tikzcd}
& B \arrow[dd, "b"] \\
A \arrow[ur, "y", "\bullet" marking] \arrow[dr, "y'"', "\bullet" marking] \arrow[r, phantom, "\theta"] & \\
& B'
\end{tikzcd}$$

4. For every $x\colon A \relbar\!\!\bullet\!\!\rightarrow B$ in P left and right identity actors

$$\begin{tikzcd}
& A \arrow[dd, "1_B"] \\
B \arrow[ur, "x", "\bullet" marking] \arrow[dr, "x"', "\bullet" marking] \arrow[r, phantom, "\lambda_x"] & \\
& B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[dr, "x", "\bullet" marking] & \\
\arrow[r, phantom, "\rho_x"] & B \\
A \arrow[uu, "1_A"] \arrow[ur, "x"', "\bullet" marking] &
\end{tikzcd}$$

A morphism of lazy profunctors

$$t\colon P \longrightarrow Q$$

$$(A \xrightarrow[P]{x} B) \longmapsto (A \xrightarrow[Q]{tx} B)$$

$$\begin{tikzcd}
A \arrow[dr, "x"] \arrow[d, phantom, "\xi" near end, shift left=4ex] & & & A \arrow[dr, "tx"] \arrow[d, phantom, "t\xi" near end, shift left=4ex] & \\
 & B & \longmapsto & & B \\
A' \arrow[uu, "a"] \arrow[ur, "x'"'] & & & A' \arrow[uu, "a"] \arrow[ur, "tx'"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \arrow[dd, "b"] & & & B \arrow[dd, "b"] \\
A \arrow[ur, "y"] \arrow[dr, "y'"'] \arrow[r, phantom, "\theta"] & & \longmapsto & A \arrow[ur, "ty"] \arrow[dr, "ty'"'] \arrow[r, phantom, "t\theta"] & \\
 & B' & & & B'
\end{tikzcd}$$

preserving the left and right identity actors.

A lazy profunctor is equivalent to a lazy category over $\mathbf{2}$, $\mathbf{P} \longrightarrow \mathbf{2}$.

Examples

For $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ we get $F_* : \mathbf{A} \longrightarrow\!\!\!\!\!\bullet\!\!\!> \mathbf{B}$

$$F_*(A,B) = \mathbf{B}(FA,B)$$

and for $U\colon \mathbf{B} \longrightarrow \mathbf{A}$ we get $U^* : \mathbf{A} \longrightarrow\!\!\!\!\!\bullet\!\!\!> \mathbf{B}$

$$U^*(A,B) = \mathbf{A}(A,UB)$$

Theorem

F is left adjoint to U if and only if $F_* \cong U^*$ i.e.,

$$\mathbf{B}(FA,B) \cong \mathbf{A}(A,UB)$$

Thank you!